FAST TRACK COMMUNICATION

Complex trajectories of a simple pendulum

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Abstract
The motion of a classical pendulum in a gravitational field of strength g is
explored. The complex trajectories as well as the real ones are determined. If g
is taken to be imaginary, the Hamiltonian that describes the pendulum becomes
$\mathcal{PT}$-symmetric. The classical motion for this $\mathcal{PT}$-symmetric Hamiltonian is
examined in detail. The complex motion of this pendulum in the presence of
an external periodic forcing term is also studied.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Many papers have been written on $\mathcal{PT}$-symmetric quantum-mechanical Hamiltonians. In
nearly all cases these Hamiltonians have had rising potentials and energies that are all discrete
and *real*. An example of such a class of Hamiltonians is [1–3]

$$H = \tfrac{1}{2}p^2 + x^2(\mathrm{i}x)^\epsilon \qquad (\epsilon \geqslant 0). \qquad (1)$$

In a few instances quantum-mechanical Hamiltonians having real continuous spectra have
been examined [4].

For the case of $\mathcal{PT}$-symmetric quantum-mechanical Hamiltonians having discrete spectra,
the associated classical-mechanical systems have also been studied [2, 5–7]. It has been
found that the classical trajectories associated with real energies are $\mathcal{PT}$-symmetric (they
are symmetric under reflection about the imaginary axis) and exhibit remarkably interesting
topological properties. These trajectories can visit multiple sheets of a Riemann surface.

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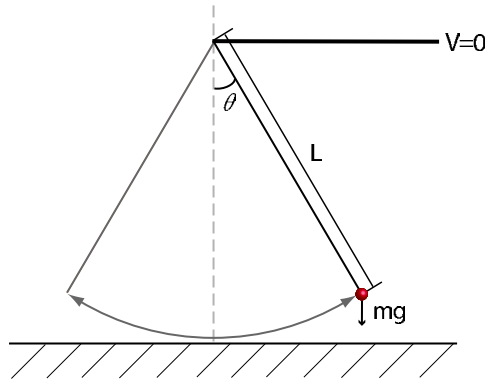
Figure 1. Configuration of a simple pendulum of mass m in a uniform gravitational field of strength g. The length of the string is L. The pendulum swings through an angle θ. We define the potential energy to be 0 at the height of the pivot.

However, the classical mechanics associated with quantum systems having continuous spectra have not yet been examined. Thus, in this paper we explore the classical trajectories of the periodic potential analyzed in [4].

Consider a classical-mechanical pendulum consisting of a bob of mass m and a string of length L in a uniform gravitational field of magnitude g (see figure 1). The gravitational potential energy of the system is defined to be zero at the height of the pivot point of the string. The pendulum bob swings through an angle θ. Therefore, the horizontal and vertical cartesian coordinates X and Y are given by $X = L \sin \theta$ and $Y = -L \cos \theta$, which gives velocities $\dot{X} = L\dot{\theta} \sin \theta$ and $\dot{Y} = -L\dot{\theta} \cos \theta$. We define the potential and kinetic energies as $V = -mgL \cos \theta$ and $T = \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2) = \frac{1}{2}mL^2\dot{\theta}^2$. The Hamiltonian $H = T + V$ for the pendulum is therefore

$$H = \tfrac{1}{2}mL^2\dot{\theta}^2 - mgL \cos \theta. \tag{2}$$

Without loss of generality we set $m = 1$ and $L = 1$, and for consistency with the notation used in earlier papers on $\mathcal{PT}$-symmetric Hamiltonians we relabel $\theta \to x$ to get

$$H = \tfrac{1}{2}p^2 - g \cos x, \tag{3}$$

where $p = \dot{x}$. The classical equations of motion for this Hamiltonian read

$$\dot{x} = \frac{\partial H}{\partial p} = p, \qquad \dot{p} = -\frac{\partial H}{\partial x} = -g \sin x. \tag{4}$$

The Hamiltonian H for this system is a constant of the motion and thus the energy E is a time-independent quantity. For most of this paper we take E to be real because the energy levels of the corresponding quantum-mechanical system are real.

This paper is organized as follows. In section 2 we study the complex trajectories for H in (3) when E and g are real. In section 3 we study the complex trajectories for H when E is real but g is imaginary. In section 4 we examine briefly the complex chaotic motion that results when the energy E of the pendulum is allowed to be complex and also when the pendulum is subject to a periodic external driving force.
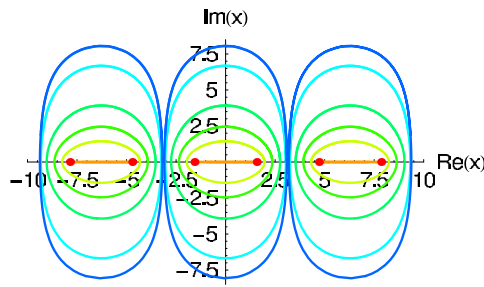
Figure 2. Complex classical trajectories for the simple pendulum in (3). These trajectories satisfy the equations of motion in (4) with $g = 1$ and $E = 0$. Note that all trajectories are closed periodic orbits and lie in cells of width 2π that entirely fill the complex plane. The usual physically observed motion of a swinging pendulum is represented by the line segments joining pairs of turning points on the real axis.

2. Pendulum in a real gravitational field

In this section we examine the solutions to Hamilton's equations (4) for the case in which the gravitational field strength g is real. Without loss of generality we take $g = 1$. There are three cases to consider: (1) the complex closed trajectories that result when $|E| < 1$, (2) the complex open trajectories that occur when $E > 1$ and (3) the complex closed trajectories that we obtain when $E < -1$.

2.1. Closed periodic trajectories: $|E| < 1$

When the energy E of a simple pendulum lies in the range $-1 \leqslant E \leqslant 1$, the physically observed motion is that of a swinging pendulum. For the example considered in this subsection we choose $E = 0$. Since a *classical turning point* x_0 is the solution to the equation $V(x_0) = E$, the classical turning points for this case satisfy $\cos x_0 = 0$. Thus,

$$x_0 = \pi/2 + n\pi \qquad (n \in \mathbb{Z}). \tag{5}$$

The observed classical trajectory for this pendulum is represented by $x(t)$ on the real-x axis. This trajectory oscillates on the real axis between the turning points at $x_0 = \pm\frac{1}{2}\pi$. However, the full array of complex solutions to the classical equations of motion is much richer (see figure 2). In this figure we see that all classical trajectories are closed and periodic and lie in cells of width 2π that fill the entire complex-x plane. As a consequence of Cauchy's theorem, all of the closed orbits have exactly the same period; namely, the period of a real physical pendulum that undergoes periodic motion on the real axis. This is because the period T for a particle of energy E for a Hamiltonian of the form $H = \frac{1}{2}p^2 + V(x)$ is given in terms of a closed complex contour integral: $T = \oint_C dx/\sqrt{2[E - V(x)]}$, where C is the complex closed path of the particle. Since the only singularity that C encloses is the square-root branch cut joining the turning points, the value of T is independent of C.

Note that the orbits in figure 2 that lie near the turning points are approximately elliptical and resemble the exactly elliptic complex trajectories of the simple harmonic oscillator, whose Hamiltonian is $H = \frac{1}{2}p^2 + \frac{1}{2}x^2$. These orbits are displayed in figure 3.

2.2. Open trajectories: $E > 1$

When the energy E of a simple pendulum is sufficiently large (here, $E > 1$), the physically observed motion is that of a rotating pendulum. The classical turning points for this case are
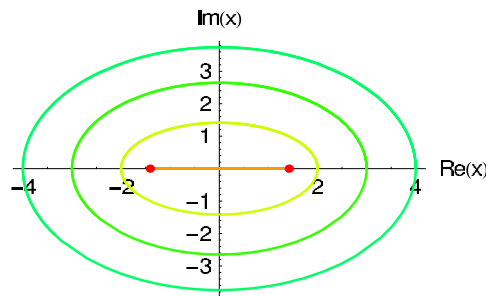
Figure 3. Complex trajectories of the simple harmonic oscillator, whose Hamiltonian is $H = \frac{1}{2}p^2 + \frac{1}{2}x^2$ with $E = 1$. These orbits are exact ellipses. These ellipses closely approximate the complex orbits in the vicinity of the turning points in figure 2.
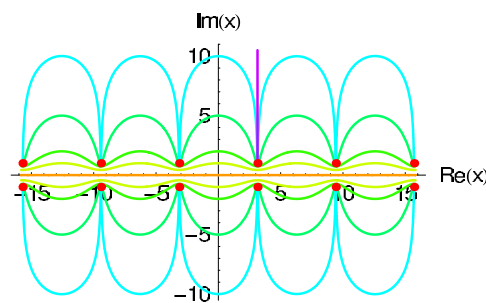

Figure 4. Trajectories in the complex-x plane for the rotating pendulum. For this case the pendulum has energy $E = \cosh 1$ and the gravitational field strength has the value $g = 1$. Note that the trajectory on the real axis corresponds to a physical pendulum undergoing continuous circular motion. All of the complex trajectories are open. The trajectories that terminate at turning points run vertically off to complex infinity. One such trajectory that starts at $x_0 = \pi + i$ is shown.

no longer on the real axis because there are no real solutions to $\cos x_0 = E$ with $E > 1$. The rotating pendulum has no real turning points because its real trajectory never turns back. To find the complex turning points, we replace x_0 by $a + ib$ and get two equations:

$$\sin a \sinh b = 0, \qquad \cos a \cosh b = -E. \tag{6}$$

For the example considered in this subsection we choose $E = \cosh 1$, and thus the turning points are situated at $a = (2k + 1)\pi$, where $k \in \mathbb{Z}$, and $b = \pm 1$.

The complex trajectories for this case are shown in figure 4. Note that the trajectories fill the entire complex plane and that there are no closed orbits. The turning points have a strong influence on the path of the complex pendulum bob. The trajectories in the upper-half complex-x plane, for example, go just below and veer around the turning points above the real axis. The turning points seem to attract the trajectories in much the same way as was found in [6].

Note that the trajectories that terminate at turning points run vertically off to complex infinity. This feature is also a property of the Hamiltonian $H = \frac{1}{2}p^2 + ix^3$, which was studied in detail in [2]. The complex trajectories for this Hamiltonian are shown in figure 5. There are three turning points, and the trajectory beginning at the turning point on the imaginary axis goes up the imaginary axis to infinity.
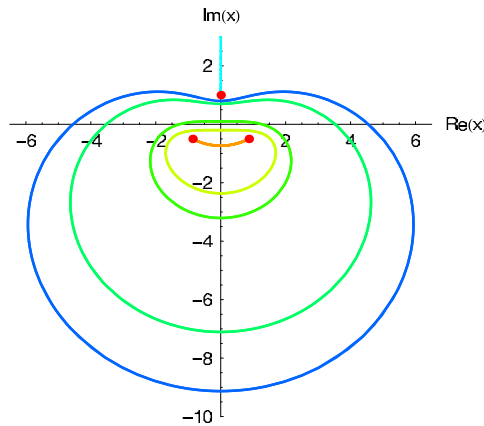
Figure 5. Complex trajectories for the Hamiltonian $H = \frac{1}{2}p^2 + ix^3$ for $E = 1$. All trajectories are closed periodic orbits except for the trajectory beginning at $x_0 = i$. A particle released from this turning point runs off to infinity in a finite time.
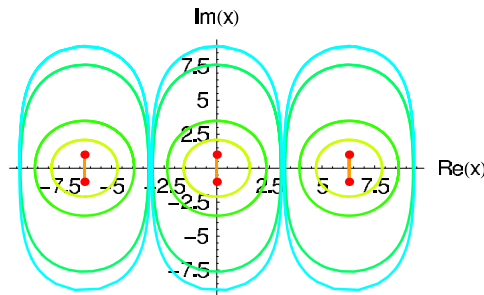

Figure 6. Complex trajectories for the pendulum in (3) with $g = 1$ and $E = -\cosh 1$. The turning points come in complex–conjugate pairs and the trajectories are all closed orbits.

One may ask how long it takes for a particle beginning at a turning point, say, the turning point at $\pi + i$, to reach infinity. The time T for this motion is represented by the following integral:

$$T = \frac{1}{\sqrt{2}} \int_{x=i+\pi}^{i\infty+\pi} \frac{dx}{\sqrt{E + \cos x}}. \tag{7}$$

A numerical evaluation of this integral gives $T = 1.975\,36\ldots$. Thus, the particle reaches the point at complex infinity in a finite time.

2.3. Closed trajectories: $E < -1$

There are no physically observable trajectories for this choice of the energy. However, the complex trajectories in this case can be calculated numerically (see figure 6) and we find that they are all closed orbits that lie in periodic cells of width 2π. The turning points come in complex–conjugate pairs. In figure 6 we have chosen $E = -\cosh 1$ and the turning points are situated at $x_0 = a + ib$, where $a = 2k\pi$ ($k \in \mathbb{Z}$) and $b = \pm 1$.
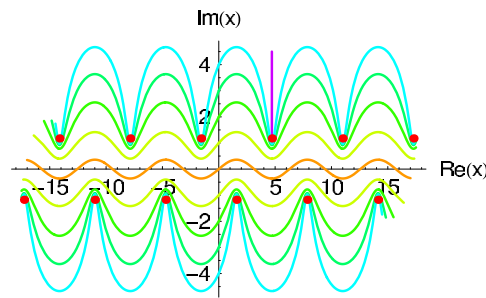
Figure 7. Complex trajectories for the pendulum in an imaginary gravitational field of strength $g = \mathrm{i}$ and positive real energy $E = \sinh 1$.
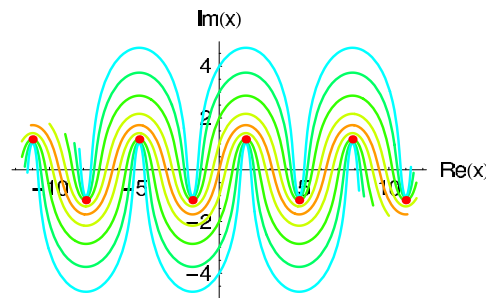

Figure 8. Complex trajectories for the pendulum in an imaginary gravitational field of strength $g = i$ and negative real energy $E = -\sinh 1$. Trajectories that begin at the turning points run off to complex infinity in the positive and negative imaginary directions as in figures 4, 5 and 7.

3. Pendulum in an imaginary gravitational field

If we take the gravitational field strength g in (3) to be imaginary, the Hamiltonian becomes $\mathcal{PT}$-symmetric, where $\mathcal{P}$ reflection consists of $x \rightarrow (2n+1)\pi - x$ and $\mathcal{T}$ reflection is complex conjugation. Setting $g = \mathrm{i}$, the Hamiltonian becomes

$$H = p^2/2 - \mathrm{i}\cos x \tag{8}$$

and the turning points are located at $x_0 = a + \mathrm{i}b$, where

$$\cos a \cosh b = 0, \qquad -\sin a \sinh b = E. \tag{9}$$

There are now two cases to consider, $E > 0$ and $E < 0$. In figure 7 we take $E = \sinh 1$ and find turning points at $x_0 = \left(n + \frac{1}{2}\right)\pi + \mathrm{i}(-1)^n$ ($n \in \mathbb{Z}$) and in figure 8 we take $E = -\sinh 1$ and find turning points at $x_0 = \left(n + \frac{1}{2}\right)\pi + \mathrm{i}(-1)^{n+1}$ ($n \in \mathbb{Z}$). Like the complex trajectories in figure 4, the trajectories in these two figures are not closed and trajectories beginning at turning points run vertically off to infinity. In figure 7 one such trajectory is shown. This trajectory begins at $\frac{3}{2}\pi + \mathrm{i}$. The time T for a particle to travel from this turning point to infinity is given by

$$T = \frac{1}{\sqrt{2}} \int_{x=\mathrm{i}+3\pi/2}^{\mathrm{i}\infty+3\pi/2} \frac{\mathrm{d}x}{\sqrt{E + \mathrm{i}\cos x}}. \tag{10}$$

After a simple change of variables this integral becomes

$$T = \frac{1}{\sqrt{2}} \int_{s=1}^{\infty} \frac{\mathrm{d}s}{\sqrt{\sinh s - \sinh 1}} = \frac{2}{\sqrt{\mathrm{e}}} K(-1/\mathrm{e}^2) = 1.845\,49\ldots, \tag{11}$$
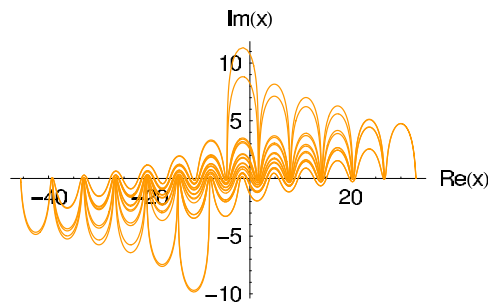
Figure 9. Complex trajectories for a pendulum in an imaginary gravitational field of strength $g = i$ and complex energy $E = i$. In this case the trajectories are not closed and not spatially periodic.
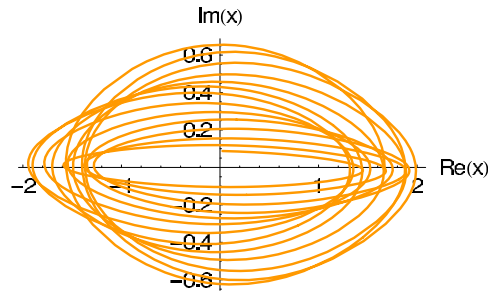

Figure 10. Complex trajectories for a pendulum in a gravitational field of strength $g = 1$ and subject to an external periodic driving force. Here, the driving force has the form $0.2\sin(0.1t)$ and the initial condition is taken to be $x = \frac{1}{2}\pi + 0.1$. Ordinarily, such an initial condition would lead to a complex trajectory like that in figure 2. For short time (up to $t = 100$ in this figure) the trajectory remains confined to the central cell in figure 2.

where K is the complete elliptic function. Thus, the classical particle reaches complex infinity in finite time.

4. Further considerations

There are a number of interesting ways to extend and generalize the investigations in the previous two sections.

In figures 7 and 8 the energy for the pendulum in an imaginary gravitational field was taken to be real. This is because a quantum-mechanical $\mathcal{PT}$-symmetric Hamiltonian has the virtue of possessing real energy levels. However, one might ask what happens if one takes the energy to be complex. In this case the complex trajectories are not spatially or temporally periodic, even though the turning points *are* periodic and are separated horizontally by 2π. The behaviour of the trajectories is illustrated in figure 9, where we have taken the energy to be $E = i$.

Another way to extend these investigations is to study what happens when there is an external periodic driving force of the form $\epsilon \sin(\omega t)$ [8–11]. We have taken $\epsilon = 0.2$ and $\omega = 0.1$ and in figures 10, 11 and 12 we have examined the complex trajectories up to times $t = 100$, $t = 650$ and $t = 1000$. The starting energy for these figures is taken to be 0 and the value of g is 1. The initial value of x is taken to be $\frac{1}{2}\pi + 0.1$, which would lead to a complex trajectory if there were no driving term. We see that for short times the trajectory is noisy, but
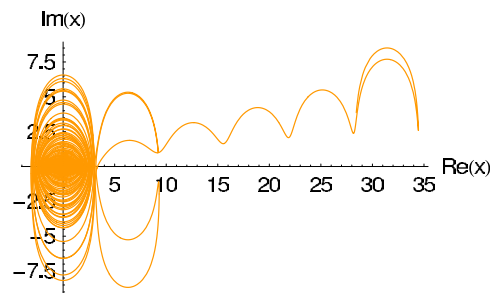
Figure 11. Same as in figure 10 but now the trajectory is allowed to continue until $t = 650$. Eventually, the trajectory escapes from its cell and begins to wander into nearby cells. It gains complex energy and it begins to resemble the trajectory in figure 9.
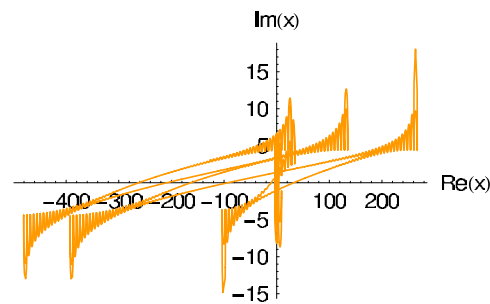

Figure 12. Same as in figure 11 but now the time runs up to $t = 1000$. Note that the complex trajectory undergoes extreme fluctuations.

seems to be confined to the central cell in figure 2. However, after a sufficiently long time the trajectory wanders into adjacent cells, as can be seen in figures 11 and 12. This result suggests it might be productive to investigate complex trajectories of dynamical systems whose real trajectories exhibit chaos, especially in the $\mathcal{PT}$-symmetric case. One could begin with the complex trajectories associated with real homoclinic chaos, as in the present case. In addition, the rich behaviour seen in the complex spatial trajectories of $\mathcal{PT}$-symmetric systems may be expected to be the projection onto the two-dimensional complex plane of interesting behaviour in four-dimensional complex phase space. For example, the phase-space trajectories of $\mathcal{PT}$-symmetric complex systems whose real parts are chaotic might exhibit a complex Hamiltonian analogue of Arnold diffusion, or some other kind of Hamiltonian diffusion.

Future progress in this topic may benefit from the application of techniques taken from the theory of elliptic curves. These techniques, which lie in the domain between complex analysis and number theory, were used to prove Fermat's last theorem [12, 13]. Physical problems suggested by the present investigation include the study of complex trajectories of the spherical pendulum, the rigid body, and the heavy top. The mapping between the simple pendulum and the rigid body [14] should provide an especially convenient approach in taking these next steps.

Acknowledgments

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